

Singletons and their maximal symmetry algebras

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Abstract

Singletons are those unitary irreducible modules of the Poincaré or (anti) de Sitter group that can be lifted to unitary modules of the conformal group. Higher-spin algebras are the corresponding realizations of the universal enveloping algebra of the conformal algebra on these modules. These objects appear in a wide variety of areas of theoretical physics: AdS/CFT correspondence, electric-magnetic duality, higher-spin multiplets, infinite-component Majorana equations, higher-derivative symmetries, etc. Singletons and higher-spin algebras are reviewed through a list of their many equivalent definitions in order to approach them from various perspectives. The focus of this introduction is on the symmetries of a singleton: its maximal algebra and the manifest realization thereof.

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1 Plan of a singleton sightseeing tour

The celebrated singletons are rather “remarkable representations”, as coined by Dirac in his seminal paper [1] on the subject. Indeed, these representations of the anti de Sitter spacetime isometry group possess several surprising properties which are so exceptional that they distinguish singletons from all other such representations. Several of these properties are reviewed here, thereby providing an elementary introduction to singletons through a list, presumably inexhaustive, of their distinct but equivalent definitions. Exhibiting the many faces of singletons could give some flavor of their ubiquitous appearances in such seemingly unrelated areas of mathematical physics as the AdS/CFT correspondence, the hydrogen atom spectrum, the electric-magnetic duality, the infinite-component Majorana equation, etc. An exhaustive bibliographical survey of the wide range of results and applications for singletons is by no means attempted here.¹ On the contrary the main focus of this short introduction is on the symmetry algebras of bosonic singletons in any dimension and on their manifest realizations. No prior knowledge of singletons is assumed, but some familiarity with the representation theory of Lie algebras is welcome. The plan is as follows:

In order to be as self-contained as possible, the isometry groups of the anti de Sitter spacetime and its conformal boundary are quickly reviewed in Section 2, as well as the corresponding representation theory classifying the elementary particles that may live on these spaces. Then comes the section 3 which presents many faces of singletons: lowest weight modules (subsection 3.1), multiplicity free modules (subsection 3.2), irreducible modules of isometry subalgebras (subsection 3.3), fields on the conformal boundary (subsection 3.4), fields on the ambient space (subsection 3.5) and kernels of the Howe dual algebra (subsection 3.6). The simplest example of singleton is the scalar one and it will serve throughout this review as a useful illustration. Finally, the section 4 reviews the various definitions of bosonic higher-spin algebras: as realizations of universal enveloping algebras (subsection 4.1), as centralizers of Howe dual algebras (subsection 4.2), as invariants of Howe dual algebras (subsection 4.3), as algebras of symmetries (subsection 4.4).

¹The bibliography has been deliberately focused either on some recent general reviews with indications of the precise location of the relevant information, or on some old seminal papers, in order to give some flavor of the early history, though from a modern viewpoint. I do apologize to the experts for the incompleteness of the bibliography.

The final message of this tour is that although higher-spin algebras can be defined in many mathematically equivalent ways, their most physical interpretation is presumably as maximal symmetry algebras of free singletons (as motivated by the AdS/CFT correspondence in the exotic strongly-curved/weakly-coupled regime).

2 Elementary particles on anti de Sitter spacetime

2.1 Anti de Sitter spacetime

The most transparent realization of AdS_{n+1} ($n \geq 1$) is via a global isometric embedding in a flat ambient space:

- The **ambient space** $\mathbb{R}^{n,2}$ is endowed with the
 - Cartesian coordinates X^A (where $A = 0, 0', 1, 2, \dots, n$) and
 - (“mostly plus”) metric $\eta_{AB} = \text{diag}(-1, -1, +1, +1, \dots, +1)$.
- The **anti de Sitter spacetime** AdS_{n+1} is the codimension one quadric (more precisely, the one-sheeted hyperboloid)

$$\eta_{AB} X^A X^B = -R^2,$$

where $R > 0$ is the curvature radius, endowed with the induced metric.

So the **isometry algebra** is manifestly the real Lie algebra

$$\mathfrak{o}(n, 2) = \text{span}_{\mathbb{R}}\{J_{AB}\}$$

which can be presented

- by its generators (the ambient “angular momenta”)

$$J_{AB} = -J_{BA} \quad (\text{where } A, B = 0, 0', 1, 2, \dots, n).$$

- modulo the commutation relations

$$[J_{AB}, J_{CD}] = i \eta_{BC} J_{AD} + \text{antisymmetrizations}.$$

and is **linearly realized** on $\mathbb{R}^{n,2}$ through the generators (the ambient “orbital angular momenta”)

$$J_{AB} = X_A P_B - X_B P_A$$

where

$$P_A = -i \frac{\partial}{\partial X^A}.$$

Usually, one of the timelike direction of $\mathbb{R}^{n,2}$, say $0'$, is particularized. Equivalently, one of the points of AdS_{n+1} , say of coordinates $X^{0'} = R$ and $X^a = 0$ (where $a = 0, 1, 2, \dots, n$), is particularized. Then the generators decompose in two sets:

- The stabilizer of $X^a = 0$, i.e. the **Lorentz subalgebra**

$$\mathfrak{o}(n, 1) = \text{span}_{\mathbb{C}}\{J_{ab}\}$$

which can be presented

- by its generators $J_{ab} = -J_{ba}$ (where $a, b = 0, 1, 2, \dots, n$)
- modulo the commutation relations

$$[J_{ab}, J_{cd}] = i \eta_{bc} J_{ad} + \text{antisymmetrizations}.$$

- The **transvections** (the displacements) generated by $\Gamma_a := R J_{0'a}$ and satisfying the commutation relations

$$[\Gamma_a, \Gamma_b] = \frac{i}{R^2} J_{ab}.$$

The transvections generators transform in the vector representation of the Lorentz subalgebra $\mathfrak{o}(n, 1)$, as can be seen in the commutation relations

$$[J_{ab}, \Gamma_c] = i \eta_{bc} \Gamma_a + \text{antisymmetrization}.$$

At the level of isometry algebras, the flat spacetime limit $AdS_{n+1} \xrightarrow{R \rightarrow \infty} \mathbb{R}^{n,1}$ translates into the Inönü-Wigner contraction $\mathfrak{o}(n, 2) \xrightarrow{R \rightarrow \infty} \mathfrak{io}(n, 1)$ where the transvections become the translations of the Poincaré algebra $\mathfrak{io}(n, 1) = \mathbb{R}^n \rtimes \mathfrak{o}(n-1, 1)$.

2.2 Conformal boundary

2.2.1 Conformal isometries of Minkowski spacetime

A **conformal metric** is an equivalence class of a metric under the equivalence relation

$$g'_{\mu\nu}(x) \sim \Omega(x) g_{\mu\nu}(x) \quad (\text{where } \mu, \nu = 0, 1, 2, \dots, n-1)$$

with $\Omega(x) > 0$ for all x^μ while a **conformal isometry** is a diffeomorphism such that the metric transforms as

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x).$$

In other words, a (conformal) isometry is a diffeomorphism that preserves the (conformal) metric. In particular, conformal isometries preserve the causal structure of spacetime (i.e. the gender of tangent vectors).

From now on, one will restrict to the case $n \geq 3$. In such case, the finite conformal isometries of the **Minkowski spacetime** $\mathbb{R}^{n-1,1}$ are generated (see e.g. [2] for a proof) by the:

- **Lorentz transformations** $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\Lambda \in O(n-1, 1)$,
- **Translations**: $x'^\mu = x^\mu + a^\mu$, where $a \in \mathbb{R}^{n-1,1}$,
- **Dilatations**: $x'^\mu = \lambda x^\mu$, where $\lambda \in \mathbb{R}_0 := \mathbb{R} - \{0\}$,

together with, either the:

- **Special conformal transformations**: $x'^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2b_\mu x^\mu + b^2 x^2}$,
where $b \in \mathbb{R}^{n-1,1}$,

or the:

- **Inversion**: $x'^\mu = \frac{x^\mu}{x^2}$.

The special conformal transformations are the most difficult to visualize but they may be understood indirectly from the following property: *Special conformal transformations are conjugate to translations, via the inversion.*

The algebra of the infinitesimal conformal isometries of $\mathbb{R}^{n-1,1}$ is $\mathfrak{o}(n, 2)$ but this is far from obvious in terms of the Cartesian coordinates x^μ . Moreover, the last two transformations (special conformal transformations and inversions) are not well defined everywhere on $\mathbb{R}^{n-1,1}$ because they map some points “at infinity” (when the denominator vanish). In order to make the conformal isometries well defined globally, it is necessary to complete the Minkowski spacetime $\mathbb{R}^{n-1,1}$ by adding “points at infinity”. The corresponding conformal compactification of $\mathbb{R}^{n-1,1}$ can be identified with the conformal boundary of AdS_{n+1} .

2.2.2 Conformal boundary of anti de Sitter spacetime

The most transparent realization of the conformal boundary ∂AdS_{n+1} of the anti de Sitter spacetime AdS_{n+1} is via its global (conformal isometric) embedding in the projectivization of the ambient space $\mathbb{R}^{n,2}$:

- The ambient space is now the projective space $\mathbb{P}(\mathbb{R}^{n,2}) \cong \mathbb{RP}^{n+1}$ endowed with the
 - **Homogeneous coordinates** X^A (where $A = 0, 0', 1, 2, \dots, n$),
 - Equivalence relation $X^A \sim \lambda X^A$ (for any $\lambda \in \mathbb{R}_0$)
 - Conformal metric (i.e. the equivalence class of) η_{AB}

As usual, points of the projective space $\mathbb{P}(\mathbb{R}^{n,2})$ are rays of $\mathbb{R}^{n,2}$.

- The **Dirac hypercone**² is the embedded codimension one quadric (more precisely, the null cone with the tip excluded)

$$\eta_{AB} X^A X^B = 0$$

quotiented by the equivalence relation, endowed with the induced conformal metric. This conformal space is the conformal boundary of the anti de Sitter spacetime AdS_{n+1} .

Geometrically, the points of ∂AdS_{n+1} are null rays of the ambient space $\mathbb{R}^{n,2}$. Topologically, the space ∂AdS_{n+1} is, essentially, homeomorphic to $S^1 \times S^{n-1}$ (where S^p denotes a hypersphere of dimension p). Heuristically, the boundary of the anti de Sitter spacetime is the asymptotic (i.e. located “at infinity”) region of intersection between the hyperboloid and the hypercone.

The **conformal isometry algebra** of the conformal boundary ∂AdS_{n+1} is manifestly the real Lie algebra $\mathfrak{o}(n, 2)$ linearly realized on $\mathbb{P}(\mathbb{R}^{n,2})$ through the (“ambient orbital angular momentum”) generators. More concretely, it is linearly realized on $\mathbb{R}^{n,2}$ through the generators $J_{AB} = X_{[A} P_{B]}$, where the square bracket denotes the antisymmetrization.

²This construction is the Euclidean analogue of the “Möbius model” in the mathematical literature. It was introduced a long time ago in physics by Dirac in a paper [3] which still remains a splendid introduction to the ambient formulation. A beautiful result of modern conformal geometry is the generalization of the ambient formulation to curved conformal spaces by Fefferman and Graham [4].

2.2.3 Conformal isometries revisited

The **light-cone coordinates**

$$X^\pm := X^{0'} \pm X^n$$

together with the **inhomogeneous coordinates**

$$x^\mu := X^\mu / X^- \quad (\text{where } \mu = 0, 1, 2, \dots, n-1)$$

provide a convenient parametrization of the Dirac hypercone in a neighborhood such that $X^- \neq 0$. Indeed, the intersection of the hypercone $-X^+X^- + \eta_{\mu\nu}X^\mu X^\nu = 0$ with a hyperplane $X^- = \text{constant} \neq 0$ is the codimension one paraboloid of equation $X^+ = X^- x^2$ parametrized by the inhomogeneous coordinates. The hyperplane $X^- = 0$ may be taken as the “hyperplane at infinity” to be added to the affine space \mathbb{R}^{n+1} in order to construct the (projective) ambient space \mathbb{RP}^{n+1} . If one identifies the conformal boundary of AdS_{n+1} with the conformal compactification of $\mathbb{R}^{n-1,1}$ then the “hyperplane at infinity” is indeed particularized.

The conformal isometries decompose as follows (see e.g. [5] for a short review):

- The ambient isometries $X'^A = \Lambda^A_B X^B$ preserving the hyperplane $X^- = \text{constant} \neq 0$, i.e. the
 - Lorentz transformations:

$$X'^+ = X^+, \quad X'^\mu = \Lambda^\mu_\nu X^\nu \quad \Longleftrightarrow \quad x'^\mu = \Lambda^\mu_\nu x^\nu$$

where $\Lambda \in O(n-1, 1)$.

- Translations:

$$X'^+ = a_\mu X^\mu, \quad X'^\mu = X^\mu + a^\mu X^- \quad \Longleftrightarrow \quad x'^\mu = x^\mu + a^\mu$$

- The ambient isometries in the plane $0'n \leftrightarrow +-$ which preserve the hyperplane at infinity $X^- = 0$, i.e. the dilatations:

$$X'^+ = \lambda X^+, \quad X'^- = \lambda^{-1} X^-, \quad X'^\mu = X^\mu \quad \Longleftrightarrow \quad x'^\mu = \lambda x^\mu$$

- The remaining transformations, i.e. the special conformal transformations:

$$X'^+ = X^+, \quad X'^- = b_\mu X^\mu, \quad X'^\mu = X^\mu + b^\mu X^+ \quad \Longleftrightarrow \quad x'^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2b_\mu x^\mu + b^2 x^2}$$

- The reflection through the hyperplane $X^n = 0$, i.e. the inversion:

$$X'^+ = X^-, \quad X'^- = X^+, \quad X'^\mu = X^\mu \quad \Longleftrightarrow \quad x'^\mu = \frac{x^\mu}{x^2}.$$

So the infinitesimal generators decompose as follows:

- The stabilizer of any hyperplane $X^- = \text{constant} \neq 0$, i.e. the **Poincaré subalgebra**

$$\mathfrak{io}(n-1, 1) = \text{span}_{\mathbb{R}}\{P_\mu, J_{\mu\nu}\} = \mathbb{R}^n \rtimes \mathfrak{o}(n-1, 1)$$

which can be presented

– by its generators

$$P_\mu := J_{+\mu}/2, \quad J_{\mu\nu} = -J_{\nu\mu} \quad (\mu, \nu = 0, 1, 2, \dots, n-1)$$

– modulo the commutation relations

$$[P_\mu, P_\nu] = 0,$$

$$[P_\mu, J_{\nu\rho}] = i\eta_{\mu\nu}P_\rho + \text{antisymmetrization},$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\eta_{\nu\rho}J_{\mu\sigma} + \text{antisymmetrizations}.$$

- The generator of ambient isometries in the plane $0'n \leftrightarrow +-$ which preserve the hyperplane at infinity, i.e. the generator of dilatation

$$\Delta := J_{+-}$$

- The remaining generators, corresponding to the infinitesimal special conformal transformations

$$K_\mu := J_{-\mu}$$

2.2.4 Distinct constant curvature spacetimes as an identical conformal space

Actually, the conformal boundary of AdS_{n+1} may be identified with any of the three constant curvature spacetimes (supplemented by “points at infinity”). These three spacetimes are geometrically realized as quadrics obtained by intersecting the hypercone with an affine hyperplane:

- **Minkowski spacetime** $\mathbb{R}^{n-1,1}$ endowed with the Cartesian coordinates x^μ as before.

- **Paraboloid:** intersection with a hyperplane orthogonal to a light-like direction, say $X^- = \text{constant} \neq 0$
- Isometry algebra: Poincaré algebra $\mathfrak{io}(\mathbf{n} - \mathbf{1}, \mathbf{1})$
- **de Sitter spacetime dS_n**
 - **Hyperboloid:** intersection with a hyperplane orthogonal to a time-like direction, say $X^{0'} = R \neq 0$
 - Isometry algebra: $\mathfrak{o}(\mathbf{n}, \mathbf{1})$
- **Anti de Sitter spacetime AdS_n**
 - **Hyperboloid:** intersection with a hyperplane orthogonal to a space-like direction, say $X^n = R \neq 0$
 - Isometry algebra: $\mathfrak{o}(\mathbf{n} - \mathbf{1}, \mathbf{2})$

The conformal compactifications of the three distinct constant curvature spacetimes $\mathbb{R}^{n-1,1}$, dS_n and AdS_n are identical: they reproduce the flat conformal space ∂AdS_{n+1} . Indeed, all of these spacetimes are conformally flat and they possess the same conformal isometry algebra $\mathfrak{o}(n, 2)$. It is important to emphasize this point because, although most of the time the conformal boundary ∂AdS_{n+1} is identified with the conformal compactification of Minkowski spacetime $\mathbb{R}^{n-1,1}$, from the point of view of conformal geometry it can equivalently be taken to be the conformal compactification of (anti) de Sitter spacetime $(A)dS_n$. This remark is useful because most results which will be mentioned here equally apply to all constantly curved spacetimes.

2.3 Unitary irreducible representations

2.3.1 Elementary particles as irreducible modules

Let \mathcal{M} be a maximally symmetric spacetime. A celebrated insight due to Wigner (see e.g. the section 2 of [6] for a review) states that there exists a one-to-one correspondence between the set of (equivalence classes of)

- (i) unitary representations of the isometry algebra of \mathcal{M} on a module (“representation space”) \mathcal{H} and
- (ii) linear relativistic wave equations describing the free propagation of a quantum particle on \mathcal{M} .

The unitary module \mathcal{H} is the Hilbert space of physical states (i.e. of inequivalent solutions of the wave equation). This identification allows

to define a **particle** as a unitary module of the spacetime isometry group. Moreover, any unitary module which is reducible is actually completely reducible. Thence, there is no loss of generality in restricting the attention to unitary irreducible modules. So one defines an **elementary particle** as an irreducible unitary module of the isometry group of the spacetime \mathcal{M} .

So the classification of the free elementary particles in the anti de Sitter spacetime $\mathcal{M} = AdS_{n+1}$ tantamounts to the classification of the irreducible unitary modules of $\mathfrak{o}(n, 2)$. Strictly speaking, the Hilbert space of physical states is usually the direct sum of two irreducible modules: the ones with either positive or negative energy corresponding respectively to the particle and its antiparticle. Therefore, if the sign of the energy is not specified in the sequel, then the direct sum of the positive energy module and its conjugate should be understood.

2.3.2 Isometry algebra of anti de Sitter spacetime

The **maximal compact subalgebra** $\mathfrak{o}(2) \oplus \mathfrak{o}(n)$ of the real Lie algebra $\mathfrak{o}(n, 2)$ corresponds to the

- **time translations** generated by the (conformal) **Hamiltonian**
 $E = J_{00}$
- **spatial rotations** generated by J_{ij} (where $i, j = 1, 2, \dots, n$)

acting in a natural way on the boundary ∂AdS_{n+1} homeomorphic to $S^1 \times S^{n-1}$. The remaining generators can be recast in the form of **ladder operators**

$$J_j^\pm = J_{0j} \mp iJ_{0'j},$$

raising or lowering the **energy** (= eigenvalue of E) by one unit. Indeed, the real Lie algebra $\mathfrak{o}(n, 2)$ can be presented equivalently

- by the generators E, J_i^\pm, J_{jk} (where $i, j, k = 1, 2, \dots, n$)
- modulo the commutation relations

$$\begin{aligned} [E, J_i^\pm] &= \pm J_i^\pm \quad ; \quad [J_{ij}, J_k^\pm] = 2i\delta_{k[j} J_{i]}^\pm \\ [J_i^-, J_j^+] &= 2(iJ_{ij} + \delta_{ij}E) \\ [J_{ij}, J_{kl}] &= i\delta_{jk}J_{il} + \text{antisymmetrizations} \end{aligned}$$

2.3.3 Orthogonal algebra

The classification of the irreducible unitary modules of $\mathfrak{o}(n, 2)$ requires the knowledge of the classification of the irreducible unitary modules of $\mathfrak{o}(n)$, which can be summarized as follows (see e.g. the section 3 of [6] for more details and references):

Irreducible unitary modules of the orthogonal group: *Let $n \geq 3$ be a positive integer and $[\frac{n}{2}]$ denote the integer part of $\frac{n}{2}$. Any unitary irreducible module $\mathcal{D}_{\ell_1, \dots, \ell_{[\frac{n}{2}]}}$ of $O(n)$ is a finite-dimensional highest (and lowest) weight module which is*

- *either tensorial or spinorial,*
- *labeled by a partition of an integer*

$$|\ell| = \ell_1 + \ell_2 + \dots + \ell_{[\frac{n}{2}]}$$

in $[\frac{n}{2}]$ parts (where $\ell_1 \geq \ell_2 \geq \dots \geq \ell_{[\frac{n}{2}]} \geq 0$).

The converse is also true.

A partition of $|\ell|$ in p parts is usually depicted as a **Young diagram** made of $|\ell|$ boxes arranged in p left-justified rows of non-increasing lengths

$$\ell_1 \geq \ell_2 \geq \dots \geq \ell_p \geq 0.$$

A renowned example of $\mathfrak{o}(n)$ -module is the **spin- s module** \mathcal{D}_s corresponding to a partition of $|\ell| = s \in \mathbb{N}$ in one part. It corresponds either to a

- tensorial module \mathcal{D}_s of $\mathfrak{o}(n)$ spanned by the components of a symmetric traceless tensor of rank s , or a
- spinorial module $\mathcal{D}_{s+1/2}$ of $\mathfrak{o}(n)$ spanned by the components of a symmetric (gamma)-traceless tensor-spinor of rank s .

2.3.4 Irreducible modules of the isometry algebra

A **Verma module** $\mathcal{V}(E_0; \ell_1, \dots, \ell_{[\frac{n}{2}]})$ of $\mathfrak{o}(n, 2)$ for E_0 positive (or negative)

- is obtained by the action of the **universal enveloping algebra** $\mathcal{U}(\mathfrak{o}(n, 2))$ on the
- **Lowest (or highest) weight vector** $|E_0; \ell_1, \dots, \ell_{[\frac{n}{2}]} \rangle$ of $\mathfrak{o}(n, 2)$, which is defined as a

- Lowest energy E_0 state

$$J_i^- |E_0; \ell_1, \dots, \ell_{[\frac{n}{2}]} \rangle = 0$$

(or highest energy state $J_i^+ |E_0; \ell_1, \dots, \ell_{[\frac{n}{2}]} \rangle = 0$)

- Lowest (and highest) weight vector of $\mathfrak{o}(n)$ labeled by

$$\ell_1 \geq \ell_2 \geq \dots \geq \ell_{[\frac{n}{2}]} \geq 0.$$

For the general definitions and properties of Verma modules, the reader may look for instance at the concise review in [7]. The ground states of energy E_0 span an irreducible finite-dimensional $\mathfrak{o}(n)$ -module labelled by the above partition.

For physical reasons, an elementary particle on anti de Sitter space-time is taken to be a positive-energy (lowest weight) unitary module, while its antiparticle is its opposite counterpart, so a negative-energy (highest weight) unitary module. Both cases can be described as **extremal weight** unitary modules.

Irreducible unitary modules of the isometry algebra: *Any extremal weight irreducible module $\mathcal{D}(E_0; \ell_1, \dots, \ell_{[\frac{n}{2}]})$ of $\mathfrak{o}(n, 2)$ is a quotient of the Verma module $\mathcal{V}(E_0; \ell_1, \dots, \ell_{[\frac{n}{2}]})$ by its maximal submodule $\mathcal{V}(E'_0; \ell'_1, \dots, \ell'_{[\frac{n}{2}]})$.*

Unitarity imposes some restrictions (see e.g. [8] and refs therein) on the possible values of the extremal energy when the ground state $\mathfrak{o}(n)$ -module carries the

- trivial representation ($\ell_1 = \ell_2 = \dots = \ell_{[\frac{n}{2}]} = 0$): either
 - * $E_0 = 0$, corresponding to the trivial $\mathfrak{o}(n, 2)$ -module, or
 - * $|E_0| \geq \frac{n}{2} - 1$ corresponding to scalar field $\mathfrak{o}(n, 2)$ -modules,
- spinor representation ($\ell_1 = \ell_2 = \dots = \ell_{[\frac{n}{2}]} = 0$):
 - * $|E_0| \geq \frac{n-1}{2}$ corresponding to the spinor field $\mathfrak{o}(n, 2)$ -modules,
- generalized spin- s representation, labeled by a Young diagram with a upper rectangle made of k rows with length $[s]$ (i.e. a partition such that $[s] = \ell_1 = \ell_2 = \dots = \ell_k > \ell_{k+1}$ with $1 \leq k \leq [n/2]$):
 - * $|E_0| \geq s + n - k - 1$ corresponding to tensorial (or spinorial) modules according to whether $2s$ is even (or odd).

The difference $\tau := |E_0| - s$ is the **twist** of such extremal weight irreducible $\mathfrak{o}(n, 2)$ -modules. The former unitarity bounds imply that $\tau \geq \frac{n}{2} - 1$.

For more details on the $n = 3$ unitary irreducible modules, an excellent pedagogical introduction is [9]. For more details on the generic construction of $\mathfrak{o}(n, 2)$ unitary irreducible modules, one may look at the appendices in [10].

3 Singletons: various definitions

3.1 Singletons as lowest weight modules

Singletons form the very exceptional subclass of the irreducible unitary modules of the algebra $\mathfrak{o}(n, 2)$ that saturate the previous unitary bound on the twist: $\tau = \frac{n}{2} - 1 = 0$. Equivalently, they saturate the unitarity bound on the extremal energy E_0 and their ground states are characterized by a rectangular Young diagram made of $[s]$ columns of height $k = \frac{n}{2}$, where s may differ from 0 or $1/2$ if and only if n is even.

The group-theoretic definition of singletons dates back to the seminal works of Dirac [1] and Flato & Frønsdal [11]. The higher-dimensional generalization³ was further developed by a variety of authors (a self-contained and rather complete treatment of the generic case can be found in [12]):

Definition: *A positive-energy **singleton** of AdS_{n+1} is a lowest weight unitary irreducible module of $\mathfrak{o}(n, 2)$ such that:*

- *When n is odd, the spin is*
 - *either $s = \frac{1}{2}$: $\mathcal{D}(\frac{n-1}{2}; \frac{1}{2})$ called **Di** or **spinor singleton***
 - *or $s = 0$: $\mathcal{D}(\frac{n}{2} - 1; 0)$ called **Rac** or **scalar singleton***
- *When n is even, the (generalized) spin s is*

³Related to the isomorphism $\mathfrak{o}(3, 2) \cong \mathfrak{sp}(4)$, the $n = 3$ singletons admit a realization in terms of bosonic oscillators [1] which is called the “metaplectic” representation of the symplectic group by mathematicians. “Singletons” are sometimes defined as the natural generalization of this construction, *c.f.* the nice review [13]. This other possibility is extremely interesting in its own but this definition is, in general, distinct from the definition that we follow here for generic n .

- any (half)integer: $\mathcal{D}(s + \frac{n}{2} - 1; [s], \dots, [s])$ called **spin s singleton** and labeled by a partition in $\frac{n}{2}$ equal parts, i.e. a rectangular Young diagram made of $\frac{n}{2}$ rows of length $[s]$.

From now on, whenever singletons of spin $s \geq 1$ will be mentioned, the integer n will always be implicitly assumed to be even, as it should.

As a nice illustration, the scalar singleton may deserve a more detailed discussion. A short review of scalar singletons can be found in [14]. The scalar singleton corresponds to the case of a lowest weight vector $|E_0; 0\rangle$ of $\mathfrak{o}(n, 2)$ annihilated by all generators of the $\mathfrak{o}(n)$ subalgebra:

$$(E - E_0)|E_0; 0\rangle = 0 \quad , \quad J_{ij}|E_0; 0\rangle = 0 \quad , \quad J_i^-|E_0; 0\rangle = 0.$$

Thus the Verma module is

$$\mathcal{V}(E_0; 0) = \text{span}_{\mathbb{R}} \{ J_{i_1}^+ \dots J_{i_s}^+ |E_0; 0\rangle \mid s \in \mathbb{N} \}$$

It can be shown (see e.g. [9] for $n = 3$) that unitarity implies $E_0 \geq \frac{n}{2} - 1$ (or $E_0 = 0$ which corresponds to the trivial representation of $\mathfrak{o}(n, 2)$). For the special value $E_0 = \frac{n}{2} - 1$ saturating the unitarity bound, the vector $\delta^{ij} J_i^+ J_j^+ |E_0; 0\rangle$ is a primitive null vector. The scalar singleton is the unitary module obtained by quotienting the maximal submodule

$$\mathcal{V}\left(\frac{n}{2} + 1; 0\right) \cong \text{span}_{\mathbb{R}} \{ \delta^{i_1 i_2} J_{i_1}^+ \dots J_{i_s}^+ |E_0; 0\rangle \mid s \in \mathbb{N} \} \subset \mathcal{V}\left(\frac{n}{2} - 1; 0\right)$$

from the Verma module. Concretely, this corresponds to factoring out the trace terms from the Verma module:

$$\begin{aligned} \mathcal{D}\left(\frac{n}{2} - 1; 0\right) &= \mathcal{V}\left(\frac{n}{2} - 1; 0\right) \Big/ \mathcal{V}\left(\frac{n}{2} + 1; 0\right) \\ &\cong \text{span}_{\mathbb{R}} \{ J_{i_1}^+ \dots J_{i_s}^+ |n/2 - 1; 0\rangle \mid \delta^{i_1 i_2} J_{i_1}^+ \dots J_{i_s}^+ |n/2 - 1; 0\rangle \sim 0 \} \end{aligned}$$

3.2 Singletons as multiplicity free modules

A nice corollary of the previous description is that under the restriction of $\mathfrak{o}(n, 2)$ to its subalgebra $\mathfrak{o}(n)$, the scalar singleton module is decomposable as follows

$$\mathcal{D}\left(\frac{n}{2} - 1; 0\right) \Big|_{\mathfrak{o}(n)} \cong \bigoplus_{s \in \mathbb{N}} \mathcal{D}_s$$

where each irreducible $\mathfrak{o}(n)$ -module \mathcal{D}_s is an eigenspace of distinct energy

$$\left(E - \left(s + \frac{n}{2} - 1\right)\right) \mathcal{D}_s = 0$$

and appears with multiplicity one. Actually, the terminology “singleton” originates⁴ from this absence of degeneracy of $\mathfrak{o}(n)$ -modules in the spectrum of the $n = 3$ “Di” and “Rac” modules, as observed initially in [15]. This property generalizes to any dimension [16].

Theorem (Angelopoulos & Laoues, 2000): *A positive energy singleton of AdS_{n+1} is a non-trivial lowest weight unitary irreducible module of $\mathfrak{o}(n, 2)$ such that its reduction to the compact orthogonal subalgebra $\mathfrak{o}(n)$ is multiplicity free. The corresponding weights of the maximal compact subalgebra $\mathfrak{o}(2) \oplus \mathfrak{o}(n)$ lie along a line in the weight diagram. More precisely,*

$$\mathcal{D}\left(s + \frac{n}{2} - 1; [s], [s], \dots, [s]\right)\Big|_{\mathfrak{o}(n)} \cong \bigoplus_{t \in \mathbb{N}} \mathcal{D}_{[s]+t, [s], \dots, [s]}$$

where each irreducible $\mathfrak{o}(n)$ -module $\mathcal{D}_{[s]+t, [s], \dots, [s]}$ is an eigenspace of distinct energy

$$\left(E - ([s] + t + \frac{n}{2} - 1)\right) \mathcal{D}_{[s]+t, [s], \dots, [s]} = 0$$

and appears with multiplicity one. Moreover, singletons are those non-trivial extremal weight unitary irreducible modules of $\mathfrak{o}(n, 2)$ such that their reduction to a compact Cartan subalgebra is multiplicity free: there is at most a single linearly independent eigenvector (i.e. a “singleton”) for each weight.

As a side remark, one may observe that the previous properties are the roots of the earliest appearances, but in disguised form, of the $n = 3$ singleton module in particle physics through the so-called infinite-component equations and the spectrum-generating algebras. These ideas were pushed forward in the sixties by the “dynamical group” research programme which revived the seminal work of Majorana [17]. A very concise account of Majorana’s publication itself and of the history of infinite-component wave equations can be found in [18]. Such constructions were recently revisited in [19] from a modern and shifted perspective (including a discussion of the higher-dimensional cases $n \geq 4$) which will be briefly reviewed now.

⁴The author thank V.K. Dobrev for pointing out to him that the name “singleton” comes from this fact.

Because of the commutation relations of the algebra $\mathfrak{o}(n, 2)$

$$[J_{ab}, J_{cd}] = i \eta_{bc} J_{ad} + \text{antisymmetrizations}$$

$$[\Gamma_a, \Gamma_b] = i J_{ab}$$

(where one has put $R = 1$ since the curvature radius has no physical interpretation in the present context), the generators Γ_a can be reinterpreted as infinite-dimensional Dirac matrices from which the “spinning” generators of $\mathfrak{o}(n-1, 1)$ are constructed as: $J_{ab} = -i \Gamma_{[a} \Gamma_{b]}$. If the wave function $\psi(x)$ on the Minkowski spacetime $\mathbb{R}^{n,1}$ takes values in the scalar singleton module $\mathcal{D}(n/2 - 1; 0)$, then the Dirac-like (i.e. the Majorana infinite-component) equation

$$(\Gamma^a P_a - M)\psi(x) = 0$$

possesses a discrete spectrum of massive solutions. As one can see by looking at this equation in the rest frame such that $(P_a - m \delta_a^0)\psi(x) = 0$, their masses m are related to their spin s by the relation

$$m = \frac{M}{s + \frac{n}{2} - 1}$$

since the eigenvalue of $\Gamma_0 = E$ on the spin- s $\mathfrak{o}(n)$ -submodule \mathcal{D}_s is equal to $s + \frac{n}{2} - 1$. Unfortunately, this “Regge trajectory” is decreasing so the infinite-component Majorana equation is physically unsatisfactory. Furthermore, tachyonic and continuous spin particles also appear in the spectrum.

As a curiosity, one may mention the following observation [19] based on the above-mentioned decomposition of the scalar singleton module: Consider an infinitely degenerate spectrum of massive particles on the flat spacetime $\mathbb{R}^{n,1}$ with equal mass for all spins (multiplicity one), i.e. a horizontal Regge trajectory. In the rest frame, the massive particle of spin s is described by the irreducible $\mathfrak{o}(n)$ -module \mathcal{D}_s , therefore the infinite tower of particles fits in an irreducible multiplet of $\mathfrak{o}(n, 2)$ corresponding to the scalar singleton module $\mathcal{D}(\frac{n}{2} - 1; 0)$. One might speculate that such an infinite multiplet could come from a highly degenerate (and exotic) spontaneous symmetry breaking of the anti de Sitter isometry algebra [19].

3.3 Singletons as irreducible modules of isometry subalgebras

The following theorem of Angelopoulos & Laoues [12] extends to any $n \geq 3$ the previous result for the case $n = 3$ of Angelopoulos, Flato, Frønsdal & Sternheimer [20].

Theorem (Angelopoulos & Laoues, 1998): *A positive (or negative) energy singleton of AdS_{n+1} is a non-trivial lowest (or highest) weight unitary irreducible $\mathfrak{o}(n, 2)$ -module that remains irreducible (or, at most, splits in two irreducible modules) under restriction to any of the following subalgebras: $\mathfrak{io}(n-1, 1)$, $\mathfrak{o}(n, 1)$ and $\mathfrak{o}(n-1, 2)$. Conversely, a singleton on ∂AdS_{n+1} is a unitary irreducible module (at most, a sum of two such modules) of any of the previous subalgebras, that can be lifted to a unitary module of $\mathfrak{o}(n, 2)$.*

Heuristically, this means that **singletons of AdS_{n+1}** are those fields:

- whose local physical degrees of freedom sit on its conformal boundary, so one may also call them **singletons on ∂AdS_{n+1}** , and
- which are preserved by the conformal symmetries in spacetime dimension n .

The absence of local degrees of freedom for singletons of AdS_{n+1} is confirmed by the fact that, upon the İnönü-Wigner contraction $\mathfrak{o}(n, 2) \rightarrow \mathfrak{io}(n, 1)$ corresponding to the flat spacetime limit $AdS_{n+1} \rightarrow \mathbb{R}^{n,1}$, they carry a trivial representation of the translation subalgebra \mathbb{R}^{n+1} in the Poincaré algebra [11].

Consequently, the extremal weight unitary irreducible $\mathfrak{o}(n, 2)$ -modules are of two types: the ones that are

- singletons on ∂AdS_{n+1} , describing **elementary particles of a CFT_n** (= n -dimensional conformal field theory), or
- *not* singletons on ∂AdS_{n+1} , describing **elementary particles in the interior of AdS_{n+1}** .

The fact that elementary particles may live either on the boundary or in the interior of anti de Sitter spacetime is the very basis of the AdS_{n+1}/CFT_n correspondence.

Let the conformal boundary ∂AdS_{n+1} be identified with the conformal compactification of $\mathbb{R}^{n-1,1}$. The following theorem was found

by Siegel in [21] but the entirely complete and rigorous proof was given later in [12].

Theorem (Siegel, 1989): *A positive-energy singleton on ∂AdS_{n+1} is a positive-energy massless unitary irreducible module of $\mathfrak{io}(n-1, 1)$ induced (à la Wigner) by a finite dimensional irreducible representation of the stabilizer $\mathfrak{io}(n-2)$ labeled by a partition in $\frac{n}{2}-1$ equal parts, i.e. by a rectangular Young diagram made of $\frac{n}{2}-1$ rows of length $[s]$.*

When seen as a representation of the Poincaré subalgebra $\mathfrak{io}(n-1, 1)$, the spin s singleton on the conformal compactification of $\mathbb{R}^{n-1,1}$ is called, for

- $n = 4$, the **helicity s representation**, and for
- higher even n and $s \geq 1$, a **spin s duality-symmetric representation** when $n/2$ is even or **chiral representation** when $n/2$ is odd, because the corresponding fieldstrength span an irreducible $\mathfrak{o}(n-1, 1)$ -module described by a rectangular Young diagram made of $\frac{n}{2}$ rows for which Hodge self-duality may be defined (more information on this point is provided in the next subsection).

Now let the conformal boundary ∂AdS_{n+1} be identified with the conformal compactification of AdS_n . The analogue of the previous theorem was obtained by Metsaev in [22] (see also [12] for a proof).

Theorem (Metsaev, 1995): *A positive-energy singleton on ∂AdS_{n+1} is a positive-energy unitary irreducible $\mathfrak{o}(n-1, 2)$ -module $\mathcal{D}(s + \frac{n}{2} - 1; [s], \dots, [s])$ saturating the unitarity bound of $\mathfrak{o}(n-1, 2)$ -modules when $s \neq 0$ (and of $\mathfrak{o}(n, 2)$ -modules when $s = 0$) and whose lowest energy $\mathfrak{o}(n-1)$ -module is labeled by a partition in $\frac{n}{2}-1$ equal parts, i.e. by a rectangular Young diagram made of $\frac{n}{2}-1$ rows of length $[s]$.*

Remark: It is important to stress that a singleton on ∂AdS_{n+1} is *not* a singleton of AdS_n , but of AdS_{n+1} . Indeed, a singleton on ∂AdS_{n+1} has lowest energy $s - 1 + n/2$ while a singleton of AdS_n has lowest energy $s - 1 + (n-1)/2$. Moreover, spin $s \geq 1$ singletons on ∂AdS_{n+1} exist only for n even while spin $s \geq 1$ singletons of AdS_n exist only for n odd.

A similar theorem holds for the conformal compactification of dS_n as well [12].

3.4 Singletons as fields on the conformal boundary

Singletons live on the conformal boundary so they can be described as fields on the corresponding compactified spacetimes. The simplest example is the scalar singleton which can be described as a massless (i.e. harmonic) scalar field $\phi(x^\mu)$ on Minkowski spacetime $\mathbb{R}^{n-1,1}$ of conformal weight $1 - \frac{n}{2}$ so that the d'Alembert equation

$$\square_{\mathbb{R}^{n-1,1}} \phi(x) = 0 \quad \text{is preserved by the conformal algebra } \mathfrak{o}(n, 2).$$

Equivalently, the scalar singleton may be described on $(A)dS_n$ through a linear wave equation involving the conformal (or Yamabe) Laplacian

$$\left(\square_{(A)dS_n} \pm \frac{n(n-2)}{4R^2} \right) \phi(x) = 0$$

where \square denotes the Laplace-Beltrami operator and the \pm symbol refer to the sign of the spacetime curvature.

The spin- s singleton $\mathfrak{io}(n-1, 1)$ -module can be realized as a space of harmonic irreducible multiforms [23] (see e.g. the section 5 of [6] for a review of the general construction of Poincaré modules). For $n = 4$, this construction reproduces the famous Bargmann-Wigner equations which are known to be conformally symmetric since a long time ago [24]. For definiteness, one will focus on tensorial singletons, i.e. integer spin $s \in \mathbb{N}$. Let θ_i^μ be a set (where $\mu, \nu = 0, 1, 2, \dots, n-1$ and $i = 1, 2, \dots, s-1, s$) of fermionic coordinates

$$\theta_i^\mu \theta_j^\nu + \theta_j^\nu \theta_i^\nu = 0,$$

on $\Pi(\mathbb{R}^{n-1,1} \otimes \mathbb{R}^s)$ (where Π reverses the Grassmann parity).

Definitions: A (differential) **multiform** on Minkowski spacetime $\mathbb{R}^{n-1,1}$ is a function $\psi(x^\mu, \theta_i^\nu)$ on the superspace $\mathbb{R}^{n-1,1} \oplus \Pi(\mathbb{R}^{n-1,1} \otimes \mathbb{R}^s)$, i.e. tensor fields on $\mathbb{R}^{n-1,1}$ with components described by a product of s columns. Moreover, a multiform is:

- **closed** if it is annihilated by all operators $d_i = \vartheta_i^\mu \frac{\partial}{\partial x^\mu}$
- **coclosed** if it is annihilated by all operators $d_i^\dagger = \frac{\partial}{\partial \vartheta_i^\mu} \frac{\partial}{\partial x^\mu}$.
- **harmonic** if it is closed and coclosed.

The differential multiforms for $s \geq 2$ generalize the usual differential forms ($s = 1$). The components of a multiform span an irreducible

$GL(n)$ -module described by a rectangular Young diagram made of s columns and $\frac{n}{2}$ rows iff it is annihilated by the operators $\theta_i^\mu \frac{\partial}{\partial \theta_i^\mu} - \delta_i^j \frac{n}{2}$ that span the algebra $\mathfrak{gl}(s)$. Moreover, it is further irreducible under $O(n-1, 1)$ iff it is also annihilated by the operators $\theta_i^\mu \theta_\mu^j$ and $\frac{\partial}{\partial \theta_i^\mu} \frac{\partial}{\partial \theta_\mu^j}$ which, together with the previous ones, span the algebra $\mathfrak{o}(2s)$.

Poincaré covariant equations for singletons [23]: *The spin- s singleton $\mathfrak{io}(n-1, 1)$ -module can be realized as a space of multiforms $\psi(x, \theta)$ on the Minkowski spacetime $\mathbb{R}^{n-1, 1}$ which are harmonic and whose components span an irreducible $\mathfrak{o}(n-1, 1)$ -module $\mathcal{D}_{[s], \dots, [s]}$ labeled by a rectangular Young diagram made of $[s]$ columns and $\frac{n}{2}$ rows.*

Remark: The multiform associated with a singleton of spin $s \geq 1$ is physically interpreted as its **fieldstrength** (or **curvature tensor**). An irreducible $O(n-1, 1)$ -module labeled by a rectangular Young diagram made of $[s]$ columns and $\frac{n}{2}$ rows decomposes a sum of two irreducible $\mathfrak{o}(n-1, 1)$ -modules when $n/2$ is odd. This subtlety is related to the involutive property of the Hodge operator. In order to treat both cases uniformly, one should consider the complexification of the $O(n-1, 1)$ -module when $n/2$ is even. Then both modules are eigenspaces (of eigenvalue ± 1) of the involutive Hodge duality, with a factor i included when $n/2$ is even (see e.g. the article [25] containing a concise introduction to the $s = 1$ case).

So the singleton modules of spin $s \geq 1$ are either said to be duality-symmetric when $n/2$ is even or chiral when $n/2$ is odd. This duality properties extend to constantly curved spacetimes, therefore the singletons of spin $s \geq 1$ are those finite-component unitary irreducible representations of one of the isometry algebras $\mathfrak{io}(n-1, 1)$, $\mathfrak{o}(n, 1)$ or $\mathfrak{o}(n-1, 2)$ which are either duality-symmetric or chiral. This deep connection between conformal symmetry and electric-magnetic duality somehow explains the appearance of singletons in many celebrated models of high-energy theoretical physics, such as maximally supersymmetric theories.

3.5 Singletons as fields on the ambient space

The main drawback of the description of singletons as fields on the conformal boundary (presented in the previous subsection) is that the conformal symmetry is not manifest (dilatation symmetry is obvious but not the special conformal and inversion symmetries). To circumvent this defect, one may describe singletons as fields on the ambient

space. Such a description was initiated by Dirac in [3] and can be summarized for the scalar singleton as follows (see e.g. the section 3 of [26] for a review of this elegant construction):

On the one hand, any space of functions of the inhomogeneous coordinates on \mathbb{RP}^{n+1} can be realized in terms of the homogeneous coordinates as a space of homogeneous functions on \mathbb{R}^{n+2} of some fixed degree. On the other hand, any space of functions on the null cone can be realized as a space of equivalence classes of functions on the ambient space modulo the functions which vanish on the null cone. The homogeneity degree is fixed by the requirement that the Laplace-Beltrami operator on the ambient space $\mathbb{R}^{n,2}$ preserves the latter equivalence relation, so that this operator induces the conformal Laplacian on the conformal boundary ∂AdS_{n+1} .

Ambient construction of the scalar singleton (Dirac, 1936): *The scalar singleton $\mathfrak{o}(n, 2)$ -module can be realized as a space of functions $\Phi(X)$ on the ambient space $\mathbb{R}^{n,2}$ which are*

- *harmonic:* $\square_{\mathbb{R}^{n,2}} \Phi(X) = 0$
- *of homogeneity degree $1 - \frac{n}{2}$:* $(X^A \partial_A + \frac{n}{2} - 1)\Phi(X) = 0$
- *quotiented by the equivalence relation*

$$\Phi(X) \sim \Phi(X) + (X^A X_A) \Xi(X)$$

where $\Xi(X)$ is of homogeneity degree $-1 - \frac{n}{2}$.

Remark: The operators \square , $X \cdot \partial_X + \frac{n+2}{2}$, X^2 are called (first class) **constraints** and they span the symplectic algebra $\mathfrak{sp}(2)$. This property will be made manifest via Howe duality. These first-class constraints find a natural interpretation in the “two-time physics” research programme of Bars (see e.g. [27]). Actually, all constraints can equivalently be imposed on the physical states.

Ambient construction of the scalar singleton: *The scalar singleton module can be realized as a space of distributions*

$$\Psi(X) := \delta(X^2)\Phi(X)$$

on ambient space $\mathbb{R}^{n,2}$ which are

- *harmonic:* $\square \Psi(X) = 0$
- *of homogeneity degree $-1 - \frac{n}{2}$:* $(X \cdot \partial + \frac{n+2}{2})\Psi(X) = 0$
- *annihilated by the quadratic form:* $X^2 \Psi(X) = 0$

The generalization of this construction to any spin [32] can be performed in the language of multiform. Let ϑ_i^A be a set (where $i = 1, 2, \dots, s-1, s$) of fermionic coordinates

$$\vartheta_i^A \vartheta_j^B + \vartheta_j^B \vartheta_i^A = 0,$$

on $\Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$.

Definitions: *An ambient multiform is*

- *a multiform on the ambient space $\mathbb{R}^{n,2}$, i.e. a function $\Psi(X^A, \vartheta_i^B)$ on the superspace $\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$.*
- *tangent to the anti de Sitter spacetime AdS_{n+1} if it is annihilated by all operators $X^A \frac{\partial}{\partial \vartheta_i^A}$.*
- *tangent to the conformal boundary $(\partial AdS)_n$ if it is annihilated by all operators $X^A \frac{\partial}{\partial \vartheta_i^A}$ and $\frac{\partial}{\partial X^A} \frac{\partial}{\partial \vartheta_i^A}$.*

The definitions of (co)closure and harmonicity for ambient multiforms are the analogues of the ones for spacetime multiforms.

Ambient construction (Arvidsson & Marnelius, 2006): The tensorial singleton $\mathfrak{o}(n, 2)$ -module can be realized as a space of multiforms $\Psi(X, \vartheta)$ on the ambient space $\mathbb{R}^{n,2}$

- which are
 - harmonic
 - of homogeneity degree $-1 - \frac{n}{2}$
 - annihilated by X^2
 - tangent to the conformal boundary
- whose components span an irreducible $\mathfrak{o}(n, 2)$ -module described by a rectangular Young diagram made of s columns and $\frac{n}{2} + 1$ rows.

This formulation is appealing because conformal symmetry is manifest, unfortunately the price to pay is that locality is not manifest any more. However, there exists a formulation [28] where both conformal invariance and locality are manifest. This is made possible by an ambient space construction in the fiber rather than in the spacetime, along the lines of the parent approach [29]. The BRST construction of [28] appears closely related to the tractor formalism [30] in conformal geometry used in the description of fields on curved ambient space (see *e.g.* [31] for some applications).

Remark: The various operators \square , $X \cdot \partial_X + \frac{n+2}{2}$, X^2 , $\frac{\partial}{\partial X} \cdot \vartheta_i$, $\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i}$, $X \cdot \frac{\partial}{\partial \vartheta_i}$, $\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i} \cdot \vartheta_i \cdot \frac{\partial}{\partial \vartheta_j} - \delta_i^j \frac{n+2}{2}$, $\vartheta_i \cdot \vartheta_j$, and $\frac{\partial}{\partial \vartheta_i} \cdot \frac{\partial}{\partial \vartheta_j}$ which annihilate the module span the orthosymplectic superalgebra $\mathfrak{osp}(2s|2)$ of constraints. This superalgebra finds a natural interpretation, on the mathematical side, in terms of Howe duality, and, on the physical side, in terms of the $\mathfrak{o}(2s)$ extended supersymmetric spinning particle (see e.g. [32, 33] and refs therein).

3.6 Singletons as kernels of the Howe dual algebra

An important message is that the orthosymplectic $\mathfrak{osp}(2s|2)$ (super)algebra of constraints annihilating the spin- s singleton module is the Howe dual of the conformal algebra $\mathfrak{o}(n, 2)$ acting on the singleton irreducible module. For a concrete description of Howe duality, one may look e.g. at the section 3 of the review [34]. For the sake of simplicity, let us turn back to the scalar singleton.

Let $T^*\mathbb{R}^{n,2}$ be the (trivial) **cotangent bundle** of the ambient space with canonical

- **Darboux coordinates** $Y_\alpha^A = (X^A, P_A)$ (where $\alpha = 1, 2$)
- **Poisson bracket**

$$\{Y_\alpha^A, Y_\beta^B\} = \varepsilon_{\alpha\beta} \eta^{AB} \iff \{X^A, P_B\} = \delta_B^A$$

where $\varepsilon_{\alpha\beta}$ is the symplectic form of $\mathfrak{sp}(2)$. The **Weyl algebra** A_{n+2} is the algebra of (polynomial) differential operators $O(X, P)$ on the ambient space $\mathbb{R}^{n,2}$, where $P_A = -i \partial / \partial X^A$. The Weyl algebra is isomorphic to the space of **Weyl symbols** $O(X, P)$, i.e. (polynomial) functions on the cotangent bundle $T^*\mathbb{R}^{n,2}$, endowed with the **Moyal star product** $*$ = $\exp i\{ , \}$.

On the one hand, the algebra $\mathfrak{o}(n, 2) = \text{span}\{L^{AB}\}$ is linearly realized on $\mathbb{R}^{n,2}$ via the generators $L^{AB} = X^A P^B - X^B P^A$ whose Weyl symbols are the bilinears

$$L^{AB} = \varepsilon^{\alpha\beta} Y_\alpha^A Y_\beta^B = X^A P^B - X^B P^A$$

On the other hand, the Lie algebra

$$\mathfrak{sp}(2) = \text{span}\{U_{\alpha\beta}\}$$

can be presented

- by its generators $U_{\alpha\beta} = U_{\beta\alpha}$ (where $\alpha, \beta = 1, 2$)
- modulo the commutation relations

$$[U_{\alpha\beta}, U_{\gamma\delta}] = i\varepsilon_{\beta\gamma}U_{\alpha\delta} + \text{symmetrizations}.$$

The Weyl symbols of the operators \square , $X \cdot \partial_X + \frac{n+2}{2}$, X^2 are the bilinears

$$U_{\alpha\beta} = \eta_{AB}Y_\alpha^A Y_\beta^B$$

In both cases, the Moyal commutators (or Poisson brackets) of generators reproduce the corresponding commutation relations. These respective realizations of $\mathfrak{o}(n, 2)$ and $\mathfrak{sp}(2)$ are maximal commutants in the algebra of quadratic Weyl symbols: they form a **Howe dual pair**.

Ambient construction of the scalar singleton: *The scalar singleton $\mathfrak{o}(n, 2)$ -module is a space of distributions $\Psi(X)$ on the ambient space $\mathbb{R}^{n,2}$ which are annihilated by the $\mathfrak{sp}(2)$ algebra Howe dual to $\mathfrak{o}(n, 2)$ in the algebra of linear operators on $\mathbb{R}^{n,2}$:*

$$U_{\alpha\beta}\Psi(X) = 0$$

The generalization to any integer spin $s \in \mathbb{N}$ is analogous [28]: The Grassmann even indices A, B will still correspond to the $(n+2)$ -dimensional ambient space $\mathbb{R}^{n,2}$ with metric η^{AB} but the letters α, β will now be superindices corresponding to a $(2|2s)$ -dimensional symplectic superspace

$$T^*\mathbb{R}^{1|s} \cong \mathbb{R}^{2|2s}$$

with symplectic form $\mathcal{J}_{\alpha\beta}$. The symplectic form on the superspace $\mathbb{R}^{2|2s}$ can be seen as a metric form on the superspace $\mathbb{R}^{2s|2} \cong \Pi(\mathbb{R}^{2|2s})$ with opposite Grassmann parity. Therefore, the symplectic form $\mathcal{J}_{\alpha\beta}$ is manifestly preserved by the orthosymplectic algebra $\mathfrak{osp}(2s|2)$. The multiforms are functions on the superspace

$$\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s) \cong \mathbb{R}^{n+2|s(n+2)}$$

with

- $n+2$ even coordinates X^A on $\mathbb{R}^{n,2}$
- $s(n+2)$ odd coordinates ϑ_i^A on $\Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$.

Let $(P_A|\pi_B^i)$ be the conjugates of the supercoordinates $(X^A|\theta_i^B)$. The phase (super)space coordinates on the cotangent bundle $T^*\mathbb{R}^{n+2|s(n+2)}$ are collectively denoted by

$$Z_\alpha^A := (X^A, P_B|\theta_i^A, \pi_B^j)$$

where the superindex α takes $2 + 2s$ values. The graded Poisson bracket originating from the symplectic structure on the phase super-space is

$$\begin{aligned} \{Z_\alpha^A, Z_\beta^B\} &= \eta^{AB} \mathcal{J}_{\alpha\beta} \\ \iff \{X^A, P_B\} &= -\{P_B, X^A\} = \delta_B^A, \quad \{\theta_i^A, \pi_B^j\} = \{\pi_B^j, \theta_i^A\} = \delta_B^A \delta_j^i. \end{aligned}$$

The phase space coordinates Z_α^A are natural coordinates on the tensor product $\mathbb{R}^{n,2} \otimes \mathbb{R}^{2|2s}$. The algebra $\mathfrak{o}(n, 2)$ is linearly realized on $\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$ as

$$J^{AB} = X^A P^B - X^B P^A - i \vartheta_i^A \frac{\partial}{\partial \vartheta_B^i} + i \vartheta_i^B \frac{\partial}{\partial \vartheta_A^i}$$

The Weyl symbols of these generators of the algebra $\mathfrak{o}(n, 2)$ are the bilinears

$$J^{AB} = \mathcal{J}^{\alpha\beta} Z_\alpha^A Z_\beta^B$$

The Lie superalgebra

$$\mathfrak{osp}(2s|2) = \text{span}\{T_{\alpha\beta}\}$$

can be presented

- by its generators $T_{\alpha\beta} = T_{\beta\alpha}$ (where $\alpha, \beta = 1, 2$)
- modulo the graded commutation relations

$$[T_{\alpha\beta}, T_{\gamma\delta}] = i \mathcal{J}_{\beta\gamma} T_{\alpha\delta} + (\text{anti})\text{symmetrizations}.$$

The Weyl symbols of the operators \square , $X \cdot \partial_X + \frac{n+2}{2}$, X^2 , $\frac{\partial}{\partial X} \cdot \vartheta_i$, $\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i}$, $X \cdot \frac{\partial}{\partial \vartheta_i}$, $\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial \vartheta_i}$, $\vartheta_i \cdot \frac{\partial}{\partial \vartheta_j} - \delta_i^j \frac{n+2}{2}$, $\vartheta_i \cdot \vartheta_j$, and $\frac{\partial}{\partial \vartheta_i} \cdot \frac{\partial}{\partial \vartheta_j}$ are the bilinears

$$T_{\alpha\beta} = \eta_{AB} Z_\alpha^A Z_\beta^B$$

The Moyal graded commutators (or graded Poisson brackets) of generators reproduce the corresponding graded commutation relations. The respective realizations of $\mathfrak{o}(n, 2)$ and $\mathfrak{osp}(2s|2)$ are maximal commutants in the algebra of quadratic Weyl symbols: they form a Howe dual pair.

Ambient construction of tensorial singletons [28]: *The spin $s \in \mathbb{N}$ singleton module $\mathcal{D}(s + \frac{n}{2} - 1; s, \dots, s)$ can be realized as a space of distributions on the superspace $\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$ which are annihilated by the $\mathfrak{osp}(2s|2)$ superalgebra, which is Howe dual to $\mathfrak{o}(n, 2)$ in the superalgebra of linear operators on $\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$.*

4 Higher-spin algebras: various definitions

4.1 Higher-spin algebras as realizations of universal enveloping algebras

The simplest higher-spin algebra is the infinite-dimensional extension of the algebra $\mathfrak{o}(n, 2)$ used by Vasiliev in his construction of a higher-spin gravity theory with AdS_{n+1} as maximally symmetric solution [35], conjectured to be the holographic dual of free scalar singletons on ∂AdS_{n+1} [36].

Definition (Vasiliev, 2003): *The quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{o}(n, 2))$ by its annihilator on the scalar singleton module $\mathcal{D}(\frac{n}{2} - 1; 0)$ is the AdS_{n+1}/CFT_n **higher-spin algebra**.⁵*

In other words, the complex AdS_{n+1}/CFT_n higher-spin algebra is the realisation of the associative algebra $\mathcal{U}(\mathfrak{o}(n + 2, \mathbb{C}))$ on the module $\mathcal{D}(\frac{n}{2} - 1; 0)$.⁶

4.2 Higher-spin algebras as centralizers of Howe dual algebras

The previous definition, based on the scalar singleton, is equivalent to a purely algebraic definition, based on the above-mentioned $\mathfrak{sp}(2)$ subalgebra of A_{n+2} (see e.g. [38] for a review of the proof).

⁵Strictly speaking, the higher-spin algebra defined in [35] is a real form of the complex algebra considered here for the sake of simplicity (see e.g. the section 5 of the review [34] for more details).

⁶Due to the realization of $n = 3$ (super)singletons in terms of oscillators (*c.f.* footnote 3), this AdS_4/CFT_3 higher-spin algebra can be defined more simply as a Weyl algebra. The higher-spin (super)algebras were originally defined as the natural generalizations of this construction [37] (see also [13]).

Theorem (Vasiliev, 2003): *The centraliser $\mathcal{C}_{A_{n+2}}(\mathfrak{sp}(2))$ of $\mathfrak{sp}(2) \subset A_{n+2}$ possesses two ideals spanned by the elements that also belong either to $\mathfrak{sp}(2)A_{n+2}$ or to $A_{n+2}\mathfrak{sp}(2)$. The quotient of the centraliser by any of these ideals is the AdS_{n+1}/CFT_n higher-spin algebra.*

4.3 Higher-spin algebras as invariants of Howe dual algebras

In order to translate the previous abstract definitions into a very concrete and explicit realization, Vasiliev made use [35] of two major contributions of Weyl: symmetric symbol calculus and classical invariant theory (see e.g. [38] for more details).

Theorem (Vasiliev, 2003): *The AdS_{n+1}/CFT_n higher-spin algebra is isomorphic to the algebra of $\mathfrak{sp}(2)$ -invariant Weyl symbols $\in A_{n+2}$ modulo $\mathfrak{o}(n, 2)$ -traces. This space is spanned by the equivalence class of polynomials that depend on the phase space variables Y_α^A only through the combination $L^{AB} = \varepsilon^{\alpha\beta} Y_\alpha^A Y_\beta^B$, modulo the polynomials which are proportional to $U_{\alpha\beta} = \eta_{AB} Y_\alpha^A Y_\beta^B$. The restriction of the adjoint representation of the AdS_{n+1}/CFT_n higher-spin algebra to the $\mathfrak{o}(n, 2)$ subalgebra is decomposable as the multiplicity free direct sum of all finite-dimensional $\mathfrak{o}(n, 2)$ -modules labeled by rectangular Young diagrams made of two rows.*

4.4 Higher-spin algebras as algebras of symmetries

The scalar singleton module $\mathcal{D}(\frac{n}{2} - 1; 0)$ may be realized as a space of distributions on $\mathbb{R}^{n,2}$ in the kernel of the representation of the algebra $\mathfrak{sp}(2) \subset A_{n+2}$, so a natural definition for its symmetries arises:

- A **symmetry generator of the scalar singleton** is an element of the centralizer of $\mathfrak{sp}(2)$ in the space A_{n+2} quotiented by the subspace $A_{n+2}\mathfrak{sp}(2)$, i.e. a differential operator \mathcal{O} on $\mathbb{R}^{n,2}$ that weakly commutes with all operators $U_{\alpha\beta}$ in the sense that:

$$[\mathcal{O} *, U_{\alpha\beta}] = P_{\alpha\beta}^{\gamma\delta} U_{\gamma\delta}$$

with $P_{\alpha\beta}^{\gamma\delta}$ some coefficients.

- A **trivial symmetry generator of the scalar singleton** is a symmetry generator that belongs to the left ideal $A_{n+2}\mathfrak{sp}(2)$, i.e. that vanishes on the kernel of the operators $U_{\alpha\beta}$.
- An **algebra of symmetries** is an algebra of equivalence classes of symmetry generators.

The equation $U_{\alpha\beta}\Psi(X) = 0$ for the distribution Ψ on $\mathbb{R}^{n,2}$ describing the scalar singleton is preserved by the transformation $\Psi \mapsto O\Psi$ when O is a symmetry generator, and $O\Psi = 0$ when the symmetry generator is trivial. The realisation of the universal enveloping algebra $\mathcal{U}(\mathfrak{o}(n, 2))$ on the scalar singleton module $\mathcal{D}(\frac{n}{2} - 1; 0)$ is automatically an algebra of such symmetry generators. Indeed, one of the leitmotiv behind higher-spin symmetries is that, while super symmetries correspond to “square roots” of isometry generators, higher symmetries are “powers” of isometry generators.

Theorem (Eastwood, 2002): *The AdS_{n+1}/CFT_n higher-spin algebra is the subalgebra of the Weyl algebra A_{n+2} of differential operators on the ambient space $\mathbb{R}^{n,2}$, spanned by (the equivalence classes of) the symmetry generators of the free scalar singleton.*

Let us finally turn back to the tensorial singletons. The spin $s \in \mathbb{N}$ singleton module $\mathcal{D}(s + \frac{n}{2} - 1; s, \dots, s)$ may be realized as a space of distributions on the superspace $\mathbb{R}^{n,2} \oplus \Pi(\mathbb{R}^{n,2} \otimes \mathbb{R}^s)$ annihilated by the $\mathfrak{osp}(2s|2)$ superalgebra, which is Howe dual to $\mathfrak{o}(n, 2)$. Following the previous track, two natural definitions arise:

- A **symmetry generator of a free spin s singleton** is an element of the centralizer of $\mathfrak{osp}(2s|2) \subset A_{n+2|s(n+2)}$ in the space $A_{n+2|s(n+2)}$ quotiented by the subspace $A_{n+2|s(n+2)}\mathfrak{osp}(2s|2)$, i.e. a differential operator that commutes with all operators $T_{\alpha\beta}$ on their kernel.
- A **trivial symmetry generator of a spin s singleton** is a symmetry generator in the subspace $A_{n+2|s(n+2)}\mathfrak{osp}(2s|2)$, so it vanishes on the singleton module.

With the help of two ingredients (the Howe duality and the Sergeev theory of classical Lie superalgebra invariants), one can determine the

Singleton maximal symmetry algebra [28]: *The following algebras are isomorphic:*

- *The maximal algebra of symmetry generators of a free spin s singleton*

- The realization of the universal enveloping algebra $\mathcal{U}(\mathfrak{o}(n+2, \mathbb{C}))$ on the spin s singleton module $\mathcal{D}(s + \frac{n}{2} - 1; s, \dots, s)$.
- The subalgebra of differential operators on the superspace $\mathbb{R}^{n+2|s(n+2)}$ whose Weyl symbols are $\mathfrak{osp}(2s|2)$ -invariants modulo $\mathfrak{o}(n+2)$ -traces.

Moreover, the adjoint representation of these algebras is a completely reducible $\mathfrak{o}(n, 2)$ -module which decomposes as the sum of all irreducible $\mathfrak{o}(n, 2)$ -modules labeled by Young diagrams such that

- (i) all columns are of even length,
- (ii) the sum of the lengths of any two columns is not greater than $n + 2$, and
- (iii) any column on the right of the $2s$ -th column is of length two,

where each such irreducible module appears with multiplicity one.

Remarks: These results

- generalize the previous works on the particular cases: any $n \geq 3$ but $s = 0$ [26, 35] or any $s \in \mathbb{N}/2$ but $n = 4$ [39]
- are formulated in terms of the above ambient definition of singleton symmetries, somewhat stronger than the local definitions adopted in [26, 28].⁷
- can be summarized in a slogan (which will be my conclusion):
All symmetry generators of a free singleton are polynomials in the isometry generators.

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⁷Indeed, for technical reasons a complete proof of the exhaustivity is lacking in [28] if one takes the local definition of singleton symmetries, but the proof is complete for the stronger definition.

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